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A generalized power function for the subsurface transmissivity profile in TOPMODEL

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Abstract. TOPMODEL, a simple physically based conceptual model, has been widely applied to various water resource investigations. Previous studies of the hydraulic transmissivity function with depth indicate that an exponential function alone may not be appropriate. Recent work suggests that a first-order hyperbolic function can provide improved simulations of base flow recession in the context of TOPMODEL. We present a generalized power function for a range of possible cases and apply this function to the TOPMODEL concepts. The specific function that is most applicable is location dependent.

1. Introduction

The development of TOPMODEL [*Beven and Kirkby*, 1979] has made major contributions to hydrological modeling. It has been widely used and applied to various water resource theories, developments, and applications [e.g., *Beven and Wood*, 1983; *Beven et al*., 1984; *Wood et al*., 1990; *Sivapalan and Wood*, 1990; *Wolock and Hornberger*, 1991; *Quinn and Beven*, 1993; *Miller and Kim*, 1996].

Soil permeability has been assumed to vary with depth as a prescribed function in the TOPMODEL concepts. In the original form of TOPMODEL [*Beven and Kirkby*, 1979] the exponential profile is assumed for the subsurface transmissivity

$$T = T_0 \exp(-\delta) \tag{1}$$

where T is the local subsurface transmissivity and T_0 is the local subsurface transmissivity at saturation. The relative storage deficit δ is defined as

$$\delta = \frac{D}{m} \tag{2}$$

where D is the local soil moisture storage deficit and m is a scaling parameter describing the decrease in T with depth.

Recent work by *Ambroise et al*. [1996] generalized the TOPMODEL concept by incorporating different transmissivity profiles within the original TOPMODEL formulation to give two other alternative forms of subsurface transmissivity profiles:

$$T = T_0(1 - \delta) \tag{3a}$$

$$T = T_0(1 - \delta)^2 \tag{3b}$$

where (3a) represents linear transmissivity and (3b) represents parabolic transmissivity with depth. These profiles are incorporated into TOPMODEL [*Ambroise et al*., 1996], resulting in an improved range of possible profiles for describing subsurface transmissivity. In this paper we present a fully generalized power function for the transmissivity profile and apply this function to the generalized TOPMODEL concepts based on *Ambroise et al*. [1996].

2. Generalized Power Function for Subsurface Transmissivity

If we assume a general power function for local subsurface transmissivity T with local soil storage deficit δ, then

$$T = T_0\left(1 - \frac{\delta}{n}\right)^n \tag{4}$$

Our nondimensional scale parameter n is assumed to be uniform throughout the basin and generalizes the solution provided by *Ambroise et al*. [1996]. From (2) the maximum storage deficit is denoted by the product of the two parameters n and m. By allowing n to range from 1 to infinity a generalized power function emerges and contains the full range of potential transmissivities (i.e., linear to exponential). For this definition to hold the ratio δ/n (or D/mn) must be less than or equal to unity. For increasing values of the scale parameter n the transmissivity profile is given as

$$\begin{aligned}
&\text{Linear} \\
&n = 1 \qquad T = T_0(1 - \delta) \\
&\text{Hyperbolic} \\
&n = 2 \qquad T = T_0\left(1 - \frac{\delta}{2}\right)^2 \\
&\text{Cubic} \\
&n = 3 \qquad T = T_0\left(1 - \frac{\delta}{3}\right)^3 \\
&\text{Exponential} \\
&n = \infty \qquad T = T_0 \exp(-\delta)
\end{aligned} \tag{5}$$

As can be seen from (4) and (5), the local transmissivity is a generalized form of the power function and can be used in TOPMODEL-related applications. By setting the scale parameter n equal to 1, 2, and infinity the linear, hyperbolic, and

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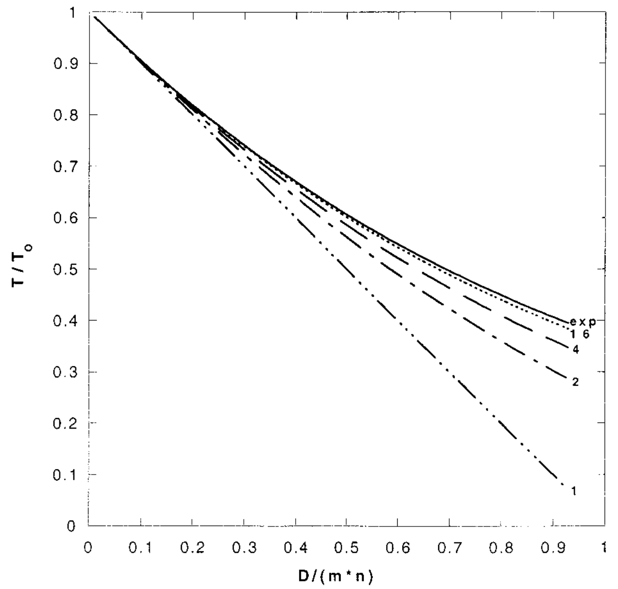

Figure 1. The normalized local subsurface transmissivity T/T_0 is plotted against D/mn (or δ/n), the ratio of the local soil storage deficit δ, and scale parameter n. The linear case is represented when $n = 1$, and the exponential case is when n approaches infinity. Values of n between 1 and 2 indicate that a large gap occurs as δ/n approaches unity.

exponential cases of the generalized power equation are derived. These three transmissivity functions have been used and are discussed in the literature [e.g., *Beven and Kirkby*, 1979; *Beven*, 1984; *Ambroise et al.*, 1996].

The shape of the transmissivity profile is controlled by the value given to the scale parameter n in (4). Figure 1 indicates that for n equal to 1 and infinity, two extremes for normalized local subsurface transmissivity T/T_0 are represented. As n increases in value from 1 to about 16, T/T_0 closely approximates the exponential profile. However, for n between 1 and 2, there is a significant gap, suggesting the need for testing a noninteger-based scale parameter in TOPMODEL applications.

3. TOPMODEL-Related Derivation

The following TOPMODEL-related derivations are based on the conceptual developments of TOPMODEL [*Beven and Kirkby*, 1970; *Ambroise et al.*, 1996]. We present here an extended generalization of TOPMODEL to the special cases of n equal to 1, 2, and infinity, as given by *Ambroise et al.* [1996].

By assuming that the local water table is parallel to the local topography and that the steady state assumption for downslope discharge can be assumed as a power function,

$$T_0 \tan \beta\left(1 - \tfrac{\delta}{n}\right)^n = aR \tag{6}$$

where a is the drainage area per unit contour for a specific location within a basin, R is the effective local recharge, and β the local slope. By rearranging (6) in terms of the effective local recharge R and defining a soil topographic index ξ [*Beven*, 1986] we have

$$R = \frac{\left(1 - \tfrac{\delta}{n}\right)^n}{\xi} \tag{7}$$

with

$$\xi = \frac{a}{T_0 \tan \beta} \tag{8}$$

Stated in terms of the relative storage deficit,

$$\delta = n(1 - \sqrt[n]{R\xi}) \tag{9}$$

and the integrated average $\bar{\delta}$ over the entire drainage A is defined as

$$\bar{\delta} = \frac{1}{A}\int_A n(1 - \sqrt[n]{R\xi}) = n\left(1 - \frac{1}{A}\int_A \sqrt[n]{R\xi}\right) \tag{10}$$

If we assume for now that the local recharge R is uniform over the drainage basin, then

$$\bar{\delta} = n\left(1 - \frac{\sqrt[n]{R}}{A}\int_A \sqrt[n]{\xi}\right) \tag{11}$$

and by substituting (7) into (11), we have

$$\bar{\delta} = n\left(1 - \frac{\left(1 - \tfrac{\delta}{n}\right)}{\sqrt[n]{\xi}}\frac{1}{A}\int_A \sqrt[n]{\xi}\right) \tag{12}$$

with

$$\gamma = \frac{1}{A}\int_A \sqrt[n]{\xi} \tag{13}$$

where γ represents the spatial average of the soil topographic index. Upon rearranging (12) we find that

$$\frac{1 - \tfrac{\bar{\delta}}{n}}{1 - \tfrac{\delta}{n}} = \frac{\gamma}{\sqrt[n]{\xi}} \tag{14}$$

Local saturation ($\delta \leq 0$) results in the inequality

$$1 - \frac{\bar{\delta}}{n} \geq \frac{\gamma}{\sqrt[n]{\xi}} \tag{15a}$$

or

$$\sqrt[n]{\xi} \geq \frac{\gamma}{1 - \tfrac{\bar{\delta}}{n}} \tag{15b}$$

Equations 15a and 15b indicate the form of the basin average saturation condition for the generalized power function for subsurface transmissivity. As shown by *Ambroise et al.* [1996], the alternative transmissivity profiles lead to the use of alternative soil topographic parameters in place of the original $\ln(a/\tan \beta)$. Here $(a/\tan \beta)^{1/n}$ can be used to map the saturation in a watershed.

The total drainage Q_b is calculated as

$$Q_b = Q_0\left(1 - \tfrac{\bar{\delta}}{n}\right)^n \tag{16}$$

where discharge at saturation, Q_0, is defined as,

$$Q_0 = \frac{A}{\gamma^n} = \frac{\sum_{j=1}^{k} l_j a_j}{\gamma^n} \tag{17}$$

and l_j and a_j are the jth channel reach contour and area within basin A, respectively. By applying the law of mass conservation,

$$Q_b = Am(d\bar{\delta}/dt) \tag{18}$$

and the chain rule we have the time derivative for base flow,

$$\frac{dQ_b}{dt} = \frac{dQ_b}{d\bar{\delta}}\frac{d\bar{\delta}}{dt} = \frac{Q_b}{Am}\frac{dQ_b}{d\bar{\delta}} \tag{19}$$

Taking the derivative of (16) with respect to the mean soil moisture deficit $\bar{\delta}$ results in

$$\frac{dQ_b}{d\bar{\delta}} = -Q_0\left(1 - \frac{\bar{\delta}}{n}\right)^{n-1} = -\frac{\mathrm{Q_b}}{\left(1 - \frac{\bar{\delta}}{\mathrm{n}}\right)} \tag{20}$$

or

$$\left(1 - \frac{\bar{\delta}}{n}\right) = \sqrt[n]{\frac{Q_b}{Q_0}} \tag{21}$$

Substituting (21) into (20) gives

$$\frac{dQ_b}{d\bar{\delta}} = -Q_b\sqrt[n]{\frac{Q_0}{Q_b}} \tag{22}$$

and inserting (22) into (19) results in

$$\frac{dQ_b}{dt} = -\frac{Q_b^2}{Am}\sqrt[n]{\frac{Q_0}{Q_b}} \tag{23a}$$

or

$$\frac{dQ_b}{Q_b^{(2-1/n)}} = -\frac{\sqrt[n]{Q_0}}{Am}\,dt \tag{23b}$$

Integrating (23b) over time period τ with a specific (or initial) discharge Q_s has the general form

$$\frac{Q_b^{1/n-1} - Q_s^{1/n-1}}{\frac{1}{n} - 1} = -\frac{Q_0^{1/n}}{Am}\tau \tag{24}$$

If we recognize that $n = 1$ is a special case [*Ambroise et al.*, 1996], we can now determine the form of the generalized base flow. Let

$$t_s = \frac{n}{n-1}\frac{Q_s^{1/n-1}}{Q_0^{1/n}}Am \tag{25}$$

$$t = t_s + \tau \tag{26}$$

then (25) simplifies to

$$Q_b = Q_s(t/t_s)^{n/(1-n)} \tag{27}$$

Figure 2 demonstrates the influence of parameter n on the normalized flow. The small value of n results in a rapid decay of flow, and the large value of n approximates an exponential form which has the slowest decay in the transmissivity profile family when other parameters are the same. In this log-log plot the slope is $n/(1 - n)$, and it asymptotically approaches -1 (the result of the exponential profile) as n becomes infinite. When transforming the recession curves to functions of time,

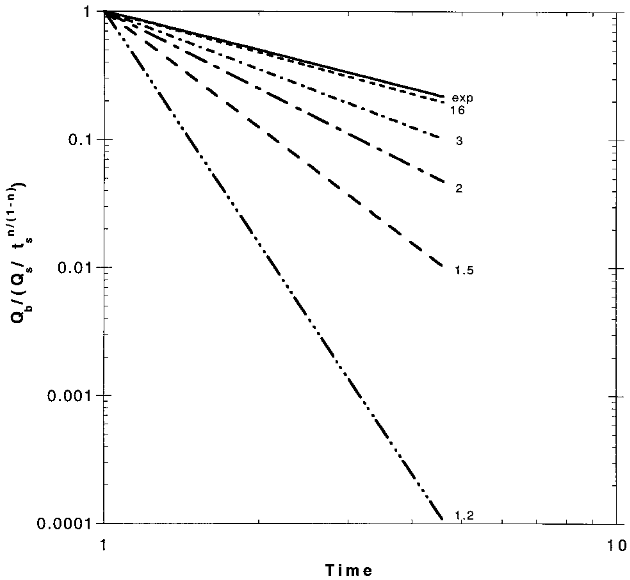

Figure 2. Normalized flow change with time using different parameter n. In this log-log plot the slope is $n/(1 - n)$, and it asymptotically approaches -1 as n becomes infinite.

$$Q_b^{1/n-1} = \alpha\tau + Q_s^{1/n-1} \tag{28}$$

and the constant slope is

$$\alpha = \left(1 - \frac{1}{n}\right)\frac{Q_0^{1/n}}{Am} = \left(1 - \frac{1}{n}\right)\frac{A^{1/n-1}}{m\gamma} \tag{29}$$

Equation 29 gives us a way to determine the scale parameter n. Terrain analyses can indicate the catchment area A and related topographic index ξ. When the type of base flow recession curve and its slope α are known, the only unknowns are the parameters m and n. If we treat mn as the maximum water deficit, a measurable quantity dependent on the soil porosity, (29) can be solved numerically for n. The modification of making mn a measurable quantity related to porosity is much more realistic than just treating it as a scale parameter. The full behavior of the range of values for n needs to be further understood. The values of n between 1 and 2 are of particular interest and are currently being studied by the authors.

4. Conclusion

A generalized power function for subsurface soil transmissivity profile has been presented. TOPMODEL-related derivations have been developed based on the generalized power function. In this derivation, previous transmissivity profiles are viewed as special cases (or subsets) of the generalized function. Further sensitivity analyses are needed to test this theory and applicability.

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